

ON A FAMILY OF MINIMUM-DRAG BODIES

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ABSTRACT

The author investigates two variational problems of supersonic gasdynamics on the determination of the shape of minimum-drag axisymmetric bodies. In the first problem, such a configuration consists of a generatrix passing through three fixed points. In the second problem, two generatrices pass through two pairs of fixed points (channel flow). The analysis is founded on the exact gasdynamic equations. The flow equations are viewed as cobonds which account for the introduction of the Lagrangian multipliers. Such a direct method of formulation of variational problems in gasdynamics was first proposed by Guderley, Armitage, and Sirazetdinov.

1. Let u, v be the x, y velocity components. The steady isentropic flows of a nonviscous non-heat-conducting gas with arbitrary thermodynamic properties is described by two equations in partial derivatives (the projected vortex and the discontinuity equation):

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$$L_1 \equiv u_y - v_x = 0, \quad L_2 \equiv (y' \rho u)_x + (y'' \rho v)_y = 0. \quad (1.1)$$

Here $v = 0$ and l , respectively, in the plane and axisymmetric cases. The density ρ , the pressure p , the speed of sound a , and the Mach angle α , are well known functions of the velocity modulus. Furthermore,

$$\frac{dp}{\rho} = a^2 \frac{d\rho}{\rho} = -u du - v dv, \quad \sin^2 \alpha = a^2 (u^2 + v^2). \quad (1.2)$$

Let the parameters of the initial isentropic, supersonic flow be prescribed (Figure 1) by the curve of the first family ae , and the shape of the profiles ab and bc is given, respectively, by equations $x = \eta_1(y)$ and $x = \eta_2(y)$. The generatrices ab and bc should be the stream lines. Hence,

$$L_1 \equiv v \eta_1'(y) - u = 0 \text{ on } ab, \quad L_2 \equiv v \eta_2'(y) - u = 0 \text{ on } bc. \quad (1.3)$$

*Numbers in the margin indicate the pagination in the original foreign text.

The coordinates of the points a, b, and c are considered as fixed. The functional, correct to the constant factor which expresses the drag of the configuration ac, is written as

$$T = \int_{v_a}^{v_b} py' dy + \int_{v_b}^{v_c} py' dy. \quad (1.4)$$

In Figure 1 the regions ae_1ea and bf_1fb represent the rarefaction flow, obtained during the flow around the convex angles in the points a and b. In the regions abf_1a and bcf_1b steady supersonic flow is obtained. The line hb represents the curve to the point b. It is assumed that within the permissible variation of the walls ab and bc, the flow mode in the characteristic triangle ace (the region under the influence of the configuration ac) remains such as it is represented in Figure 1.

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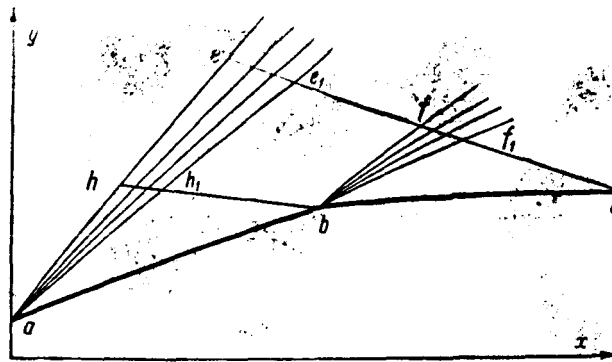


Figure 1

Let us formulate the following variational problem: in the initial flow prescribed by the curve ae define the generatrices ab and bc , passing through the prescribed points a , b , and c and realizing the extremum of the functional (1.4) with the differential bonds (1.3) on ab and bc and the differential bonds (1.1), (1.2) in the region ace .

Let us designate the flow region $hbceh$ by τ_2 , and the region $abha$ by τ_1 . In view of the fact that the Equations (1.1) employed are hyperbolic, the variation of the flow parameters in the region τ_2 affects the pressure distribution only along the wall bc , therefore the differential bonds (1.1) in the region τ_2 will be considered by means of the factors $h_{12}(x, y)$, $h_{22}(x, y)$, and in the region τ_1 by means of the factors $h_{11}(x, y)$, $h_{21}(x, y)$. Let the factors $c_1(y)$, $c_2(y)$, respectively, account for the bonds (1.3):

Let us compose the functional:

$$T^0 = \int_{y_a}^{y_b} [py' + c_1(y) l_1] dy + \int_{y_b}^{y_c} [py' + c_1(y) l_2] dy + \\ + \iint_{\tau_1} (L_1 h_{11} + L_2 h_{21}) dx dy + \iint_{\tau_2} (L_1 h_{12} + L_2 h_{22}) dx dy.$$

In the permissible variation, the variations T and T^0 agree in view of Equations (1.1) - (1.3). Let us determine the functions $u(x, y)$, $v(x, y)$, $\eta_1(y)$, and $\eta_2(y)$ so that the functional T^0 attains an extremal value. Here p and ρ are functions of u and v , and, according to Formula (1.2)

$$\delta p = -\rho u \delta u - \rho v \delta v, \quad \delta \rho = -\frac{\rho u}{a^2} \delta u - \frac{\rho v}{a^2} \delta v.$$

The variations of double integrals coupled by the changes in the boundaries in the regions τ_i ($i = 1$ and 2), are absent due to the equality to zero of the integrands. Since the points a , b , and c are fixed, then the variations of the integrals along ab and bc , related to the changes in the coordinates in the points a , b , and c , are also absent. The derivatives of the variations of the functions are excluded in the double integrals by means of the formulas of integration by parts and Green's.

Taking the above into account, and utilizing Equations (1.1) - (1.3), we find /1030

$$\delta T = \delta T^0 = \int_{y_a}^{y_b} (U_1 \delta u + V_1 \delta v + W_1 \delta \eta_1) dy + \int_{y_b}^{y_c} (U_2 \delta u + V_2 \delta v + W_2 \delta \eta_2) dy + \\ + \iint_{\tau_1} (U_{31} \delta u + V_{31} \delta v) dx dy + \iint_{\tau_2} (U_{32} \delta u + V_{32} \delta v) dx dy + \\ + \int_{y_c}^{y_e} (U_4 \delta u + V_4 \delta v) dy + \int_{y_h}^{y_b} (U_{52} - U_{51}) \delta u + (V_{52} - V_{51}) \delta v] dy.$$

The factors with the variations of u , v , η_1 , and η_2 are the well known functions of flow parameters and Lagrangian multipliers. The integral along the prescribed curve ae vanishes, since on it $\delta u = \delta v = 0$. Let us determine the Lagrangian multiplier, converting δT^0 to zero.

In the regions τ_1 , we have

$$U_{31} \equiv -(h_{11})_y + y^* \rho \frac{uv}{a^2} (h_{21})_y - y^* \rho \left(1 - \frac{u^2}{a^2}\right) (h_{21})_x = 0, \\ V_{31} \equiv (h_{11})_x + y^* \rho \frac{uv}{a^2} (h_{21})_x - y^* \rho \left(1 - \frac{v^2}{a^2}\right) (h_{21})_y = 0. \quad (1.5)$$

Equating the coefficients with $\delta\eta_1$ and $\delta\eta_2$ to zero, we have

$$c_1(y) = -y^{\rho}(u + \lambda_1) \text{ on } ab, \quad c_2(y) = -y^{\rho}(u + \lambda_2) \text{ on } bc,$$

where λ_1 and λ_2 are arbitrary constants of integration.

Equating to zero the coefficients in the integrals along the generatrices with δu and δv , and taking into account the above formulas, we have

$$\begin{aligned} h_{11} &= -y^{\rho}v, \quad h_{12} = -(u + \lambda_1) \text{ on } ab, \\ h_{13} &= -y^{\rho}v, \quad h_{22} = -(u + \lambda_2) \text{ on } bc. \end{aligned} \quad (1.6)$$

Equating to zero the coefficients with δu and δv in the integrals along the curves of the second family ec and hb , we obtain, after simple transformations,

$$h_{12} - h_{22} y^{\rho} \operatorname{ctn} \alpha = 0 \text{ on } ec, \quad (1.7)$$

$$h_{13} - h_{11} - (h_{22} - h_{21}) y^{\rho} \operatorname{ctn} \alpha = 0 \text{ on } hb. \quad (1.8)$$

In satisfying the conditions (1.5) - (1.8), the first variation of the functional (1.4) becomes zero.

Let us perform an analysis of the obtained equations and boundary conditions for the determination of the functions $h_{11}(x, y)$, $h_{21}(x, y)$. The equation system (1.5) at supersonic velocities is of the hyperbolic type. The equations of the curves are of the form:

$$dy = \operatorname{tg}(\theta + \alpha) dx, \quad dh_{11} - dh_{21} y^{\rho} \operatorname{ctn} \alpha = 0 \text{ (first family)} \quad (1.9)$$

$$dy = \operatorname{tg}(\theta - \alpha) dx, \quad dh_{11} + dh_{21} y^{\rho} \operatorname{ctn} \alpha = 0 \text{ (second family)} \quad (1.10)$$

Here θ is the angle of inclination of the velocity vector to axis x .

The differential stipulation of consistency (1.10) on the curve ec may be integrated by means of condition (1.7). We have

$$h_{12} = (\lambda_3 y^{\rho} \operatorname{ctn} \alpha)^{1/2}, \quad h_{22} = (\lambda_3 / y^{\rho} \operatorname{ctn} \alpha)^{1/2}, \quad (1.11)$$

where λ_3 is the arbitrary constant of integration.

The values of the functions h_{11} , h_{21} on ab , and h_{21} , h_{22} on bc , definable by formulas (1.6), are initial data in the solution of the Cauchy problem for the system (1.5). They define the functions h_{11} , h_{21} in the region abh_1a , and the function h_{12} , h_{22} in the region bcb_1b . A result obtained in the work [3] shows that the equalities (1.6) are integrals of the system (1.5) and thus

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define the functions h_{1i} , h_{2i} in the regions indicated above. Employing the integrals for h_{1i} , and h_{2i} and the conditions (1.11), (1.8) we have

$$u - v \tan \alpha = -\lambda_2, \quad y^* \rho v^2 \tan \alpha = \lambda_3 \text{ on } f_1 c, \quad (1.12)$$

$$E \equiv h_{11} + y^* \rho v - (h_{22} + u + \lambda_1) y^* \rho \sin \alpha = 0 \text{ on } h_1 b, \quad (1.13)$$

$$h_{12} = -y^* \rho v, \quad h_{22} = -(u + \lambda_2) \text{ on } b f_1, \quad (1.14)$$

$$h_{11} = -y^* \rho v, \quad h_{21} = -(u + \lambda_1) \text{ on } a h_1. \quad (1.15)$$

The relations (1.12) were obtained by Yu. D. Shmyglevskiy [4] in an investigation of the variational problem in the construction of the generatrix with fixed points b and c and with the fixed curve bf . The constants λ_2 and λ_3 may be determined from the value of the gasdynamic parameters in the point f_1 . The solution of the Gurs problem for equations (1.5) according to the values of h_{12} and h_{22} on the curves bf_1 and $f_1 e_1$ (formulas (1.14) and (1.11)) define h_{12} and h_{22} in the region $h_1 b f_1 e_1 h$. The constant λ_1 is determined from the conditions of vanishing of the function E in the point b on approaching the point b from the side of the wall ab .

The condition (1.8) on the segment $h h_1$, the formulas (1.11) and (1.15), and the values obtained for h_{12} , h_{22} on the segment $h_1 e_1$, may be used to determine the factors h_{1i} , h_{2i} in the region $a e_1 e a$. If, furthermore, the function E on the segment $h_1 b$ becomes zero and the relations (1.12) are satisfied on the curve $f_1 c$, then the variation of the functional (1.4) on the constructed flow will vanish. The following fact should also be noted. The curve hb is, generally speaking, the discontinuity line for the Lagrangian multipliers which account for the differential bonds (1.1). In this respect, the book by N. M. Gyunter [5] should be mentioned. For variational problems in gasdynamics this fact was first indicated by A. N. Krayko [6].

The analysis performed for the conditions of stationarity makes possible the construction of a solution by means of the iteration process. On the prescribed curve ae the point e is fixed, and, furthermore, the angle of deflection of the configuration in the point b is also fixed. The interval from x_a to x_b is divided into N segments with fixed abscissas, x_n ($n = 0, 1, \dots, N$). A generatrix ab , which is the initial approximation of the sought generatrix, is drawn through the points a and b . A solution by the numerical method of characteristics of the streamline problem of the generatrix ab provides the flow parameters in the region $ab f_1 e a$. This makes it possible to calculate the constants λ_3 and λ_2 in the point f_1 . The knowledge of the value of the factors h_{12} , h_{22} on the curves $e_1 f_1$ and $f_1 b$, makes it possible, by solving the Gurs problem by the numerical method of characteristics, to determine the factors in the region $h_1 b f_1 e_1 h_1$, to calculate the constant λ_1 , and to determine the function E on the curve $h_1 b$. The angles of inclination (x_n) of the new generatrix ab are found by means of formula

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$$\vartheta_{j+1}(x_n) = \vartheta_j(x_n) + \varepsilon E_j[\xi(x_n)] + \vartheta_{j+1}^0.$$

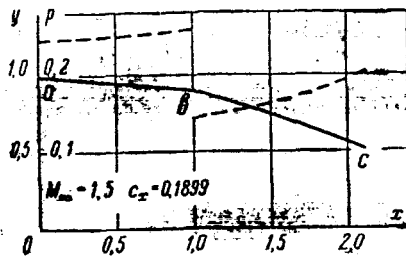


Figure 2

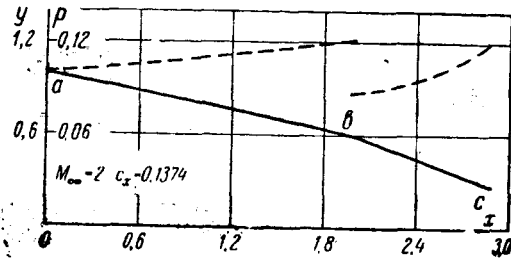


Figure 3

Here j is the number of iterations, $\xi(x_n)$ is the number of curves of the first family resulting from the point of the generatrix ab with the abscissa x_n , ε is a random number, and $|\varepsilon| < 1$, ϑ_{j+1}^0 is the parameter which defines the conditions for the new generatrix to pass through the points a and b . The calculations may be repeated until the function E on h_1b is equal to zero. After this, the curve f_1c is drawn from the point f_1 to the point c with a zero flow rate value. The generatrix bc is found from the solution by the numerical method of characteristics of the Gurs problem according to the curves bf_1 and f_1c which are now known.

The constructed configuration ac is the sought configuration if the prescribed point c coincides with that obtained. Two arbitrary parameters (the position of the point e on the prescribed curve ae and the angle of deflection of the configuration in the point b) make it possible to solve the linear problem.

As an example, a calculation is made of the rear generatrices of minimum-drag bodies of revolution. The supersonic incoming flow of an ideal gas with an adiabatic curve index of $\chi = 1.4$ is assumed to be unperturbed. Figures 2 and 3 show the configurations passing through the prescribed points a , b , and c , and with a minimum drag for Mach numbers $M_\infty = 1.5$ and $M_\infty = 2$. The broken lines represent the distribution along the pressure generatrices p , related to $\rho_\infty u_\infty^2$. The values of the drag coefficients c_x are given in the figures.

2. Let us examine the problem of the determination of the shape of the generatrices passing through the fixed points a , b and c , d , and ensuring minimum channel drag. Figure 4 shows one of the possible schemes of optimum flow. The initial flow, prescribed by the curve ae , is considered to be isentropic and nonvortical. The region of influence of $abfdcea$ of the sought configuration is bounded by the generatrices of the channel, the prescribed curve ae , and the curves passing through the points c , d , and b . In Figure 4, the flow region af_1ff_2cea represents the region of the interaction of rarefaction waves generated by the deflections of the stream lines in the points a and c . The regions abf_1a and cf_2dc represent steady supersonic flow. As in the first section, the flow in the region of influence of the configuration is assumed to be isentropic. Furthermore, it is assumed that within the permissible variation of the channel generatrices the flow scheme remains such

as it is represented in Figure 4. It will be shown that this scheme may provide a solution to the variational problem in minimum-drag channels. The functional which describes the configuration drag is written as

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$$T = \int_{v_a}^{v_b} p y^* dy + \int_{v_d}^{v_c} p y^* dy. \quad (2.1)$$

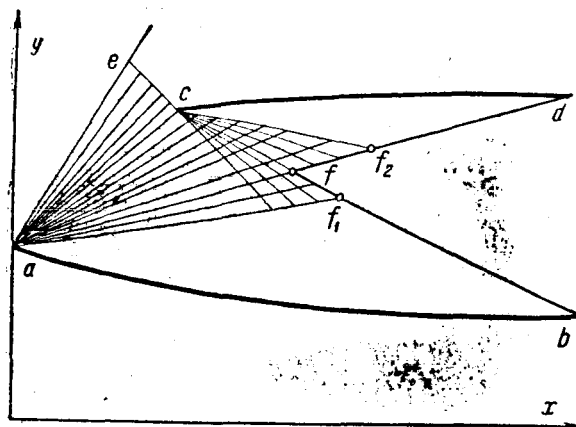


Figure 4

Let the shape of the profiles ab and cd , respectively, be prescribed by the equations $x = \eta_1(y)$, $x = \eta_2(y)$. The generatrices ab and cd should be the stream lines. Hence, equations (1.3) are valid along them.

Let us formulate the following variational problem: in the initial flow prescribed by the curve ae to define the generatrices ab and cd , which pass through the prescribed points a , b and c , d and which realize the extremum of the functional (2.1) with the differential bonds (1.3) on ab and cd and the differential bonds (1.1), (1.2) in the region of influence of τ .

Let us divide the region of influence of the sought configuration into four subregions, and designate τ_1 as the subregion $abfa$, τ_2 as the subregion $cfdc$, τ_3 as the subregion $afca$, and τ_4 as the subregion $acea$. In composing the Lagrangian functional T^0 the bonds (1.1) in the subregions are τ_i ($i = 1, 2, 3, 4$). The Lagrangian multipliers $h_{1i}(x, y)$, $h_{2i}(x, y)$ are accounted for by means of the four groups. The bonds (1.3) on the generatrices ab and cd are taken into account, respectively, by means of the multipliers $c_1(y)$, $c_2(y)$. Then

$$T^0 = \int_{v_a}^{v_b} [p y^* + c_1(y) l_1] dy + \int_{v_d}^{v_c} [p y^* + c_2(y) l_2] dy + \sum_{i=1}^4 \iint_{\tau_i} (L_1 h_{1i} + L_2 h_{2i}) dx dy.$$

The calculation of the variations of the functional T^0 is analogous to the calculations performed in Section 1. The conditions of stationarity are of

the following form: in the subregion τ_1 for the determination of the factors h_{11} , h_{21} the equations (1.5) are valid, and in the generatrices ab and cd the formulas (1.6) are obtained. Further, we have

$$\begin{aligned} h_{14} - h_{24} y^* \rho \operatorname{ctn} \alpha &= 0 && \text{on } ec, \\ h_{11} - h_{21} y^* \rho \operatorname{ctn} \alpha &= 0 && \text{on } fb, \\ h_{13} + h_{22} y^* \rho \operatorname{ctn} \alpha &= 0 && \text{on } fd, \\ h_{13} - h_{12} - (h_{23} - h_{22}) y^* \rho \operatorname{ctn} \alpha &= 0, && \text{on } fc, \\ h_{13} - h_{11} + (h_{23} - h_{21}) y^* \rho \operatorname{ctn} \alpha &= 0, && \text{on } fa, \\ h_{14} - h_{13} + (h_{24} - h_{23}) y^* \rho \operatorname{ctn} \alpha &= 0, && \text{on } ca. \end{aligned}$$

By employing the integrals of the multipliers h_{11} , h_{21} in the region abf_1a , the integrals of the multipliers h_{12} , h_{22} in the region cf_2dc and the conditions of consistency (1.9), (1.10) which were integrated by means of the written out conditions, it is possible to demonstrate that the obtained conditions of stationarity will be satisfied provided the following relations are satisfied:

$$u - v \tan \alpha = -\lambda_1, \quad y^* \rho v^2 \tan \alpha = \lambda_3 \quad \text{on } f_1 b, \quad (2.2)$$

$$u + v \tan \alpha = -\lambda_2, \quad y^* \rho v^2 \tan \alpha = \lambda_4 \quad \text{on } f_2 d, \quad (2.3)$$

where λ_1 , λ_3 and λ_2 , λ_4 are arbitrary constants of integration definable from the values of the gasdynamic parameters in the points f_1 and f_2 , respectively.

The construction of the optimal channel is achieved as follows: the flow field in the region af_1ff_2cea is constructed by the numerical method of characteristics on the prescribed curve ae and the point c . By means of formulas (2.2) and (2.3) curves are plotted from the points f_1 and f_2 to the points b and d with a flow rate value of $\psi = 0$ and $\psi = \psi_c$. Further, using the known curves af_1 , f_1b , and cf_2 , f_2d the generatrices ab and cd are found by solving the Gurs problem. Four arbitrary lines relative to the position of the points f_1 and f_2 in the pencils of rarefaction make it possible, generally speaking, to plot the extremal curves f_1b and f_2d to the prescribed points b and d .

As an example, axisymmetric optimal channels were constructed with generatrices passing through the prescribed points a , b and c , d . The incoming supersonic flow of an ideal gas with an adiabatic curve index of $\eta = 1.4$ was assumed to be unperturbed. Figures 5 and 6 show minimum-drag channels for Mach number $M_\infty = 1.5$. The broken lines represent the distribution along the pressure generatrices p , related to $\rho_\infty u_\infty^2$. The values of the drag coefficients c_x are equal to: $c_x = 0.1748$ and $c_x = 0.2728$, respectively. In calculating c_x an area of a cross section with a radius y_a was employed.

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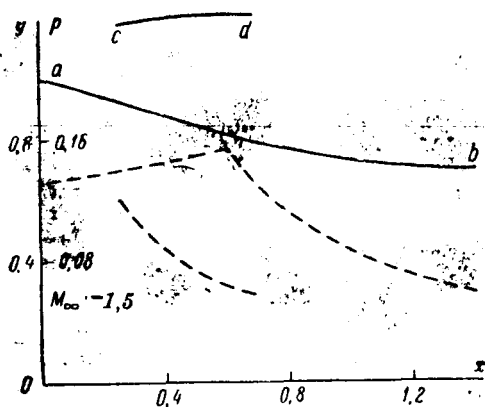


Figure 5

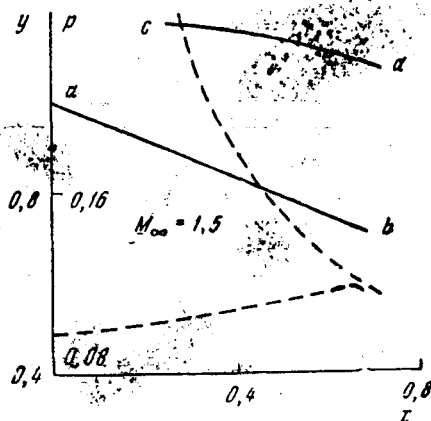


Figure 6

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